CSE 4392 Special Topics
Natural Language Processing

Expectation Maximization

2024 Spring

Intuition of EM

- Let's say I have 3 coins in my pocket,
  - Coin 0 has probability λ of heads
  - Coin 1 has probability p_1 of heads
  - Coin 2 has probability p_2 of heads
- **For each trial:**
  - First, I toss Coin 0
  - If coin 0 turns up **heads**, I toss coin 1 three times
  - If coin 0 turns up **tails**, I toss coin 2 three times
  - I don't tell you the results of the coin 0 toss, or whether coin 1 or coin 2 was tossed, but I tell you how many heads/tails are seen after each trial
- • You see the following sequence:

$$\langle H, H, H\rangle, \langle T, T, T\rangle, \langle H, H, H\rangle, \langle T, T, T\rangle, \langle H, H, H\rangle$$

Quiz: Guess what are the estimated values of λ, p_1, p_2?

Maximal Likelihood Estimate

- Data points $x_1, x_2, \ldots, x_n$ from (finite or countable) set $\mathcal{X}$ (x_i is a triplet of three tosses)
- Parameter vector θ
- Parameter space Ω
- We have a distribution $P(x \mid \theta)$ for any $\theta \in \Omega$, such that

$$\sum_{x \in \mathcal{X}} P(x \mid \theta) = 1$$

$$P(x \mid \theta) \geq 0, \forall x$$

- Assume data points are drawn independently and identically distributed from a distribution $P(x \mid \theta^*)$ for some $\theta^* \in \Omega$

Log Likelihood

- Probability distribution $P\ (x \mid \theta)$ for any $\theta \in \Omega$
- Likelihood of θ:

$$Likelihood(\theta) = P(x_1, x_2, \ldots, x_n \mid \theta) = \prod_{i=1}^{n} P(x_i \mid \theta)$$

- Log likelihood of θ:

$$L(\theta) = \sum_{i=1}^{n} \log P(x_i \mid \theta)$$

Example 1: Coin Tossing

- $\mathcal{X} = \{H, T\}$. Our data set $x_1, x_2, \ldots, x_n$ is a sequence of heads and tails, e.g.,

  HTHTHHHTTT

- Parameter vector θ is a single parameter, i.e. probability of coin showing heads
- Parameter space $\Omega = [0, 1]$
- Distribution $$P(x \mid \theta) = \begin{cases} \theta \text{ if } x = H \\ 1 - \theta \text{ if } x = T \end{cases}$$

Example 2: Markov Chains

- $\mathcal{X}$ is the set of all possible state (or tag) sequences generated by an underlying generative process. Our sample is n sequences $X_1, X_2, \dots, X_n$, where $X_i \in \mathcal{X}$.
- θ_T is the vector of all transition $(s_i \rightarrow s_j)$ parameters. W.L.O.G., we assume there is a dummy start state ϕ and initial transtion $\phi \rightarrow s_1$
- Let $T(\alpha) \subset T$ be all transtion of the form $\alpha \rightarrow \beta$
- Ω is the set of $\theta \in [0,1]^{|S+1||S|}$ where S is the set of all states (tags), such that:

$$\forall \alpha \in S, \sum_{t \in T(\alpha)} \theta_t = 1$$

Example 2: Markov Chains

- Since θ_T is the vector of all transtion parameters
- We have:

$$P(X\,|\theta_T) = \prod_{t \in T} \theta_t^{Count(X,t)}$$

where *Count(X, t)* is the number of times transition *t* occures in sequence *X*.

- This gives:

$$\log(P(X|\theta_T) = \sum_{t \in T} Count(X,t) \log \theta_t$$

$$L(\theta_T) = \sum_i \log P(X_i|\,\theta_T) = \sum_i \sum_{t \in T} Count\,(X,t) \log \theta_t$$

MLE for Markov Chains

- We use θ for θ_T for simplicity
- To solve for $\theta_{MLE} = \arg\max_{\theta \in \Omega} L(\theta)$
- We solve θ in

$$\frac{\partial L(\theta)}{\partial \theta} = 0$$

with appropritate probability constraints

- Therefore:

$$\theta_t = \frac{\sum_i Count(X_i, t)}{\sum_i \sum_{t' \in T(\alpha)} Count(X_i, t')}$$

where t is a transition of the form $\alpha \rightarrow \beta$ for some β, $T(\alpha)$ is all the transitions originating from α.

Models with Hidden Variables

- Suppose we have two sets $\mathcal{X}$ and $\mathcal{Y}$, and a joint distribution $P(x, y \mid \theta)$
- If we have **fully-observable data**, (x_i, y_i) pairs, then

$$L(\theta) = \sum_i \log P(x_i, y_i \mid \theta)$$

- If we have **partially-observable data**, x_i examples only, then

$$L(\theta) = \sum_i \log P(x_i \mid \theta)$$
$$= \sum_i \log \sum_{y \in \mathcal{Y}} P(x_i, y \mid \theta)$$

- This is unsupervised learning, very similar to clustering.
- We will use *an interative algorithm* to infer θ like k-means

EXPECTATION-MAXIMILATION

- If we have **partially-observable data**, x_i examples only, then

$$L(\theta) = \sum_i \log \sum_{y \in \mathcal{Y}} P(x_i, y \mid \theta)$$

- The EM (Expectation Maximization) algorthm is a method for finding

$$\theta_{MLE} = \arg\max_{\theta} L(\theta) = \arg\max_{\theta} \sum_i \log \sum_{y \in \mathcal{Y}} P(x_i, y \mid \theta)$$

Three Coins Example

- In the three-coin example:
  - $\mathcal{Y} = \{H, T\}$ (possible outcomes of coin 0)
  - $\mathcal{X} = \{HHH, TTT, HTT, THH, HHT, TTH, HTH, THT\}$
  - $\theta = \{\lambda, p_1, p_2\}$
- And $P(x, y \mid \theta) = P(y \mid \theta)\, P(x \mid y, \theta)$

where

$$P(y|\theta) = \begin{cases} \lambda \text{ if } y = H \\ 1 - \lambda \text{ if } y = T \end{cases}$$

and

$$P(x|y,\theta) = \begin{cases} p_1^h\,(1-p_1)^t \text{ if } y = H \\ p_2^h\,(1-p_2)^t \text{ if } y = T \end{cases}$$

h is num of heads in x
t is num of tails in x

Three Coins Example

- Calculate various probabilities:

one H and two T from THT

$$P(x = THT, y = H \mid \theta) = \lambda p_1 (1 - p_1)^2$$

$$P(x = THT, y = T \mid \theta) = (1 - \lambda) p_2 (1 - p_2)^2$$

$$\begin{aligned} P(x = THT \mid \theta) &= P(x = THT, y = H \mid \theta) + P(x = THT, y = T \mid \theta) \\ &= \lambda p_1 (1 - p_1)^2 + (1 - \lambda) p_2 (1 - p_2)^2 \end{aligned}$$

$$\begin{aligned} P(y = H \mid x = THT, \theta) &= \frac{P(x = THT, y = H \mid \theta)}{P(x = THT \mid \theta)} \quad \text{(Bayes rule)} \\ &= \frac{\lambda p_1 (1 - p_1)^2}{\lambda p_1 (1 - p_1)^2 + (1 - \lambda) p_2 (1 - p_2)^2} \end{aligned}$$

THREE COINS EXAMPLE

- Suppose fully observed data looks like:

$$(\langle HHH\rangle, H), (\langle TTT\rangle, T), (\langle HHH\rangle, H), (\langle TTT\rangle, T), (\langle HHH\rangle, H)$$

- In this case, the maximum likelihood estimates of the parameters are:

$$\lambda = \frac{3}{5}$$
$$p_1 = \frac{9}{9} = 1$$
$$p_2 = \frac{0}{6} = 0$$

THREE COINS EXAMPLE

- Partial observed data might look like:

$$\langle HHH \rangle, \langle TTT \rangle, \langle HHH \rangle, \langle TTT \rangle, \langle HHH \rangle$$

- How do you estimate the MLE parameters?

Three Coins Example

- Partial observed data might look like:

$$\langle HHH\rangle, \langle TTT\rangle, \langle HHH\rangle, \langle TTT\rangle, \langle HHH\rangle$$

- If the current parameters are λ, p_1, p_2

$$P(y = H \mid x = \langle HHH\rangle) = \frac{P(\langle HHH\rangle, H)}{P(\langle HHH\rangle, H) + P(\langle HHH\rangle, T)}$$

$$= \frac{\lambda p_1^3}{\lambda p_1^3 + (1-\lambda)p_2^3}$$

$$P(y = H \mid x = \langle TTT\rangle) = \frac{P(\langle TTT\rangle, H)}{P(\langle TTT\rangle, H) + P(\langle TTT\rangle, T)}$$

$$= \frac{\lambda(1-p_1)^3}{\lambda(1-p_1)^3 + (1-\lambda)(1-p_2)^3}$$

Three Coins Example

- If the current parameters are λ, p_1, p_2

$$P(y = H \mid x = \langle HHH \rangle) = \frac{P(\langle HHH \rangle, H)}{P(\langle HHH \rangle, H) + P(\langle HHH \rangle, T)}$$

$$= \frac{\lambda p_1^3}{\lambda p_1^3 + (1-\lambda) p_2^3}$$

$$P(y = H \mid x = \langle TTT \rangle) = \frac{P(\langle TTT \rangle, H)}{P(\langle TTT \rangle, H) + P(\langle TTT \rangle, T)}$$

$$= \frac{\lambda (1-p_1)^3}{\lambda (1-p_1)^3 + (1-\lambda)(1-p_2)^3}$$

- If λ=0.3, $p_1 = 0.3$, $p_2 = 0.6$

$$P(y = H \mid x = \langle HHH \rangle) = 0.0508$$

$$P(y = H \mid x = \langle TTT \rangle) = 0.6967$$

Three Coins Example

- After filling in hidden variables for each example, the partially observed data looks like this:

$$(\langle \mathrm{HHH} \rangle, H) \quad P(y = \mathrm{H} \mid \mathrm{HHH}) = 0.0508$$
$$(\langle \mathrm{HHH} \rangle, T) \quad P(y = \mathrm{T} \mid \mathrm{HHH}) = 0.9492$$

sum to 1

$$(\langle \mathrm{TTT} \rangle, H) \quad P(y = \mathrm{H} \mid \mathrm{TTT}) = 0.6967$$
$$(\langle \mathrm{TTT} \rangle, T) \quad P(y = \mathrm{T} \mid \mathrm{TTT}) = 0.3033$$

sum to 1

$$(\langle \mathrm{HHH} \rangle, H) \quad P(y = \mathrm{H} \mid \mathrm{HHH}) = 0.0508$$
$$(\langle \mathrm{HHH} \rangle, T) \quad P(y = \mathrm{T} \mid \mathrm{HHH}) = 0.9492$$

sum to 1

$$(\langle \mathrm{TTT} \rangle, H) \quad P(y = \mathrm{H} \mid \mathrm{TTT}) = 0.6967$$
$$(\langle \mathrm{TTT} \rangle, T) \quad P(y = \mathrm{T} \mid \mathrm{TTT}) = 0.3033$$

sum to 1

$$(\langle \mathrm{HHH} \rangle, H) \quad P(y = \mathrm{H} \mid \mathrm{HHH}) = 0.0508$$
$$(\langle \mathrm{HHH} \rangle, T) \quad P(y = \mathrm{T} \mid \mathrm{HHH}) = 0.9492$$

sum to 1

Three Coins Example

- New estimates:

$$\langle HHH\rangle, \langle TTT\rangle, \langle HHH\rangle, \langle TTT\rangle, \langle HHH\rangle$$

$$\begin{array}{ll}
(\langle \mathrm{HHH}\rangle, H) & P(y=\mathrm{H} \mid \mathrm{HHH}) = 0.0508 \\
(\langle \mathrm{HHH}\rangle, T) & P(y=\mathrm{T} \mid \mathrm{HHH}) = 0.9492 \\
(\langle \mathrm{TTT}\rangle, H) & P(y=\mathrm{H} \mid \mathrm{TTT}) = 0.6967 \\
(\langle \mathrm{TTT}\rangle, T) & P(y=\mathrm{T} \mid \mathrm{TTT}) = 0.3033 \\
& \ldots
\end{array}$$

out of 5 coin 0 tosses how may are heads?

$$\lambda = \frac{3 \times 0.0508 + 2 \times 0.6967}{5} = 0.3092$$

how many heads in X_i?

$$p_1 = \frac{3 \times 3 \times 0.0508 + 0 \times 2 \times 0.6967}{3 \times 3 \times 0.0508 + 3 \times 2 \times 0.6967} = 0.0987$$

$$p_2 = \frac{3 \times 3 \times 0.9492 + 0 \times 2 \times 0.3033}{3 \times 3 \times 0.9492 + 3 \times 2 \times 0.3033} = 0.8244$$

Summary of Three Coins Example

- Begins with λ=0.3, $p_1 = 0.3$, $p_2 = 0.6$

- Fill in hidden variables using:

$$P(y = H \mid x = \langle HHH \rangle) = 0.0508$$

$$P(y = H \mid x = \langle TTT \rangle) = 0.6967$$

- Re-estimate parameters to be

$$\lambda=0.3092,\ p_1 = 0.0987,\ p_2 = 0.8244$$

EM Interations

$P(y = H \mid X_i)$

Iteration	λ	p_1	p_2	$\tilde{p}_1$	$\tilde{p}_2$	$\tilde{p}_3$	$\tilde{p}_4$	$\tilde{p}_5$
0	0.3000	0.3000	0.6000	0.0508	0.6967	0.0508	0.6967	0.0508
1	0.3092	0.0987	0.8244	0.0008	0.9837	0.0008	0.9837	0.0008
2	0.3940	0.0012	0.9893	0.0000	1.0000	0.0000	1.0000	0.0000
3	0.4000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000

- Coin example for $\{\langle HHH\rangle, \langle TTT\rangle, \langle HHH\rangle, \langle TTT\rangle, \langle HHH\rangle\}$
- λ is now 0.4, indicating that coin 0 has a probability 0.4 of selecting the tail-biased coin (coin 1)
- θ (parameters) are like the cluster centers in k-means

EM INTERATIONS

Iteration	λ	p_1	p_2	$\tilde{p}_1$	$\tilde{p}_2$	$\tilde{p}_3$	$\tilde{p}_4$
0	0.3000	0.3000	0.6000	0.0508	0.6967	0.0508	0.6967
1	0.3738	0.0680	0.7578	0.0004	0.9714	0.0004	0.9714
2	0.4859	0.0004	0.9722	0.0000	1.0000	0.0000	1.0000
3	0.5000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000

- Coin example for $x = \{\langle HHH \rangle, \langle TTT \rangle, \langle HHH \rangle, \langle TTT \rangle\}$.
- This solution of $\lambda = 0.5, p_1 = 0, and\ p_2 = 1$ is intuitively correct: the coin tosser has two coins, one which always shows heads, and another which always shows tails, and is picking between them with equal probability .
- Posterior probabilities $\bar{p_i}$ show that we are certain that coin 1 (tail-biased) generate x_2 and x_4, whereas coin 2 generated x_1 and x_3.

Initialization Matters

Iteration	λ	p_1	p_2	$\tilde{p}_1$	$\tilde{p}_2$	$\tilde{p}_3$	$\tilde{p}_4$
0	0.3000	0.7000	0.7000	0.3000	0.3000	0.3000	0.3000
1	0.3000	0.5000	0.5000	0.3000	0.3000	0.3000	0.3000
2	0.3000	0.5000	0.5000	0.3000	0.3000	0.3000	0.3000
3	0.3000	0.5000	0.5000	0.3000	0.3000	0.3000	0.3000
4	0.3000	0.5000	0.5000	0.3000	0.3000	0.3000	0.3000
5	0.3000	0.5000	0.5000	0.3000	0.3000	0.3000	0.3000
6	0.3000	0.5000	0.5000	0.3000	0.3000	0.3000	0.3000

- Coin example for $x = \{\langle HHH\rangle, \langle TTT\rangle, \langle HHH\rangle, \langle TTT\rangle\}$.
- In this case, EM is stuck in a "**saddle point**", or ***local optimal***.

INTIALIZATION MATTERS

Iteration	λ	p_1	p_2	$\tilde{p}_1$	$\tilde{p}_2$	$\tilde{p}_3$	$\tilde{p}_4$
0	0.3000	0.7001	0.7000	0.3001	0.2998	0.3001	0.2998
1	0.2999	0.5003	0.4999	0.3004	0.2995	0.3004	0.2995
2	0.2999	0.5008	0.4997	0.3013	0.2986	0.3013	0.2986
3	0.2999	0.5023	0.4990	0.3040	0.2959	0.3040	0.2959
4	0.3000	0.5068	0.4971	0.3122	0.2879	0.3122	0.2879
5	0.3000	0.5202	0.4913	0.3373	0.2645	0.3373	0.2645
6	0.3009	0.5605	0.4740	0.4157	0.2007	0.4157	0.2007
7	0.3082	0.6744	0.4223	0.6447	0.0739	0.6447	0.0739
8	0.3593	0.8972	0.2773	0.9500	0.0016	0.9500	0.0016
9	0.4758	0.9983	0.0477	0.9999	0.0000	0.9999	0.0000
10	0.4999	1.0000	0.0001	1.0000	0.0000	1.0000	0.0000
11	0.5000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000

Coin example for $x = \{\langle HHH \rangle, \langle TTT \rangle, \langle HHH \rangle, \langle TTT \rangle\}$.

- Just modify p_1 a bit, EM is able to skip the saddle point and reach global optimum.

THE EM ALGORTHM

- θ^t is the parameter vector at the t^{th} iteration.
- Choose θ^0 at random (or using some smart heuristics)
- Iterative procedure defined as:

$$\theta^t = \arg\max_{\theta} Q(\theta, \theta^{t-1})$$

where

$$Q(\theta, \theta^{t-1}) = \sum_{i} \sum_{y \in \mathcal{Y}} P(y \mid x_i, \theta^{t-1}) \ \log P(x_i, y \mid \theta)$$

The EM Algorithm

- (E-step): Compute *expected* counts.

$$\overline{Count}(r) = \sum_{i=1}^{n} \sum_{y} P(y \mid x_i, \theta^{t-1})\ Count(x_i, y, r)$$

for every paramter θ_r, e.g.,

$$\overline{Count}(DT \rightarrow NN) = \sum_{i} \sum_{y} P(S \mid O_i, \theta^{t-1})\ Count(O_i, S, \theta_{DT \rightarrow NN})$$

- (M-step): Re-estimate parameters using expected counts to *maximize* likelihood.

e.g.,

$$\theta_{DT \rightarrow NN} = \frac{\overline{Count}(DT \rightarrow NN)}{\sum_{\beta} \overline{Count}(DT \rightarrow \beta)}$$

The EM Algorithm

- Intuition: Fill in hidden variables according to $P(y \mid x_i, \theta)$
- EM is guaranteed to converge to a local maximum, or saddle-point, of the likelihood function
- In general, if

$$\arg\max_{\theta} \sum_{i} \log P(x_i, y_i \mid \theta)$$

has a simple analytic solution, then

$$\arg\max_{\theta} \sum_{i} \sum_{y} P(y \mid x_i, \theta) \log P(x_i, y \mid \theta)$$

also has a simple solution.

Example: EM for HMM

- We observe only word sequences $X_1, X_2, \ldots, X_n$ (no tags)
- θ is the vector of all transition parameters (include initial state distribution as a special case, $\phi \rightarrow s$
- ϕ is the vector of all emission parameters
- Initialize parameters θ^0 and ϕ^0

Example: EM for HMM

- Initialize parameters θ^0 and ϕ^0
- E-step:

θ_k has nothing to do with X_i

$$\overline{Count}(\theta_k) = \sum_{i=1}^{n} \sum_{Y} P(Y|X_i, \theta^{t-1}, \phi^{t-1})\ Count(X_i, Y, \theta_k)$$

$$= \sum_{i=1}^{n} \sum_{Y} P(Y|X_i, \theta^{t-1}, \phi^{t-1})\ Count(Y, \theta_k)$$

$$\overline{Count}(\phi_k) = \sum_{i=1}^{n} \sum_{Y} P(Y|X_i, \theta^{t-1}, \phi^{t-1})\ Count(X_i, Y, \phi_k)$$

Example: EM for HMM

- M-step:

$$\theta_k^t = \frac{\overline{Count}(\theta_k)}{\sum_{\theta' \in M(\theta_k)} \overline{Count}(\theta')}$$

where $M(\theta_k)$ is the set of all transitions ($a \to b$, for all b) that share the same previous state as the k^{th} transition ($a \to c$, for some c)

$$\phi_k^t = \frac{\overline{Count}(\phi_k)}{\sum_{\phi' \in M'(\phi_k)} \overline{Count}(\phi')}$$

where $M'(\phi_k)$ is the set of all emissions ($a \to x$, for all x) that share the same previous state as the k^{th} emission ($a \to x'$, for some x').

Efficient EM?

- E-step:

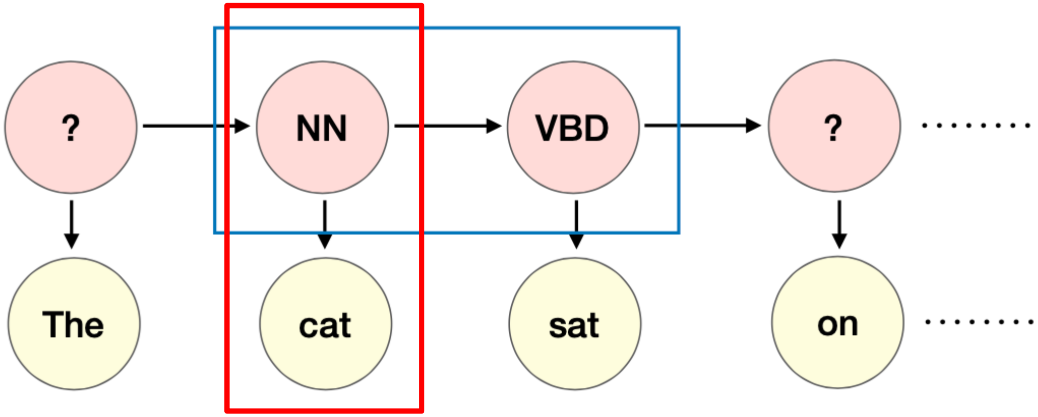
$$\overline{Count}(\theta_k) = \sum_{i=1}^{n} \sum_{Y} P(Y \mid X_i, \theta^{t-1}, \phi^{t-1})\ Count(Y, \theta_k)$$

$$\overline{Count}(\phi_k) = \sum_{i=1}^{n} \sum_{Y} P(Y \mid X_i, \theta^{t-1}, \phi^{t-1})\ Count(X_i, Y, \phi_k)$$

- Can't enumerate all possible Y's!

Quiz: How many possible Y's are there? Assume your own parameters before computing the answer.

Efficient EM?



- E-step:

$$\overline{Count}(\theta_{NN\to VBD}) = \sum_{i=1}^{n}\sum_{Y} P(Y|X_i,\theta^{t-1},\phi^{t-1})\ Count(Y,\theta_k)$$

$$= \sum_{i}\sum_{j=1}^{m} \boxed{P(y_j = NN, y_{j+1} = VBD|X_i,\theta^{t-1},\phi^{t-1})}$$

where m is the length of sequence X_i.

Similary, $\overline{Count}(\phi_{NN\to cat}) = \sum_{i}\sum_{j:X_{ij}=cat} P(y_j = NN|X_i,\theta^{t-1},\phi^{t-1})$

Forward-Backward Algorithm

- Define:

$$\alpha_s(j) = P\left(x_1, \ldots, x_{j-1}, y_j = s \,\middle|\, \theta, \phi\right) \text{ (forward probability)}$$

$$\beta_s(j) = P\left(x_j, \ldots, x_m \middle| y_j = s, \theta, \phi\right) \text{ (backward probability)}$$

- Observation likelihood:

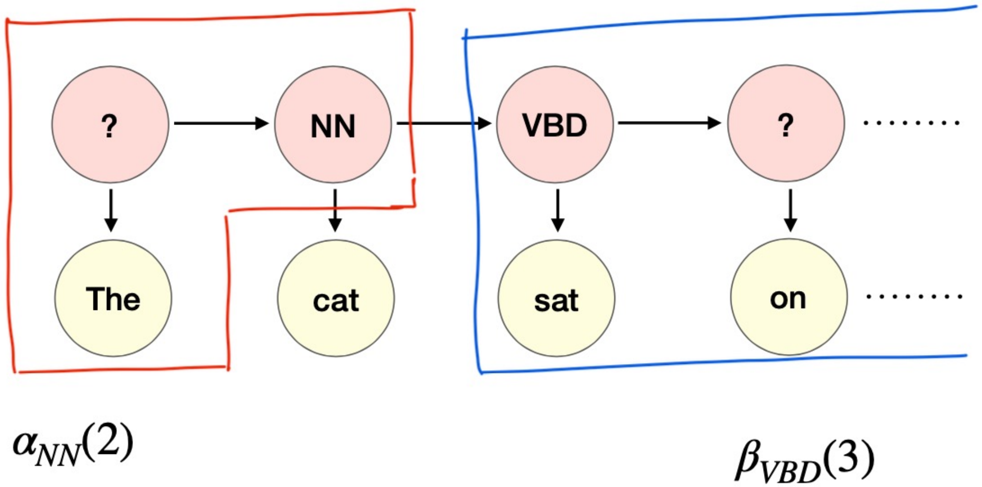
$$Z = P(x_1, \ldots, x_m | \theta, \phi) = \sum_s \alpha_s(j)\beta_s(j) \;\; \forall j \in 1, \ldots, m$$

- Thus,

$$P(y_j = s \,|\, X, \theta, \phi) = \frac{\alpha_s(j)\beta_s(j)}{Z}$$

$$P(y_j = s, y_{j+1} = s' \,|\, X, \theta, \phi) = \frac{\alpha_s(j)\; \theta_{s \to s'}\; \phi_{s \to x_j}\; \beta_{s'}\,(j+1)}{Z}$$

α AND β



$$P(y_j = s \mid X, \theta, \phi) = \frac{\alpha_s(j)\beta_s(j)}{Z}$$

$$P(y_j = s, y_{j+1} = s' \mid X, \theta, \phi) = \frac{\alpha_s(j)\ \theta_{s\to s'}\ \phi_{s\to x_j}\ \beta_{s'}\ (j+1)}{Z}$$

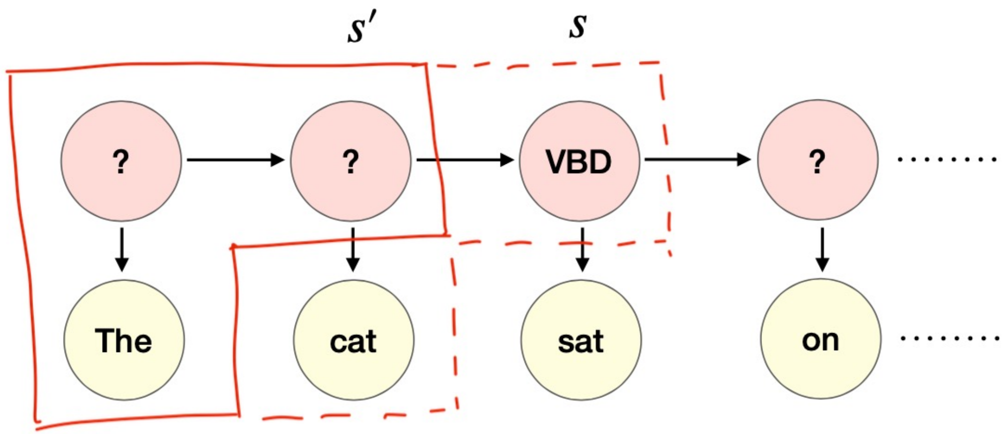
Now we can estimate:

$$\overline{Count}(\theta_{s\to s'}) = \sum_i \sum_{j=1}^{m} P(y_j = s, y_{j+1} = s' \mid X_i, \theta, \phi)$$

$$\overline{Count}(\phi_{s\to o}) = \sum_i \sum_{j:X_{ij} = o} P(y_j = s \mid X_i, \theta, \phi)$$

Dynamic Programming

$$\begin{aligned}\alpha_s(j) &= P(y_j = s, x_1, \ldots, x_{j-1}) \\ &= \sum_{s'} P(y_{j-1} = s', x_1, \ldots, x_{j-2})\ P(x_{j-1} \,|\, y_{j-1} = s')\ P(y_j = s \,|\, y_{j-1} = s') \\ &= \sum_{s'} \alpha_{s'}\ (j-1)\ \phi_{s' \to x_{j-1}}\ \theta_{s' \to s}\end{aligned}$$



Dynamic Programming

$$\begin{aligned}\alpha_s(j) &= P(y_j = s, x_1, \ldots, x_{j-1}) \\ &= \sum_{s'} P(y_{j-1} = s', x_1, \ldots, x_{j-2})\ P(x_{j-1} \,|\, y_{j-1} = s')\ P(y_j = s \,|\, y_{j-1} = s') \\ &= \sum_{s'} \alpha_{s'}\ (j-1)\ \phi_{s' \to x_{j-1}}\ \theta_{s' \to s}\end{aligned}$$

Similarly,

$$\beta_s(j) = \phi_{s \to x_j}\ \sum_{s'} \beta_{s'}\ (j+1)\ \ \theta_{s \to s'}$$

Time complexity: $O(|S|^2 \cdot m)$